Topic: Applied optimization

Question: What dimensions require the least amount of fencing?

A rancher plans to build a rectangular fenced area adjacent to an existing stone wall. The rancher wants the fence to enclose an area of $160{,}000$ square meters for his horses, but is low on fencing materials.

Answer choices:

A 400×400

B 283×565

C $4{,}000 \times 8{,}000$

D $100 \times 1{,}600$

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Solution: B

Since one side of the area does not need any fencing, the amount of fencing needed to enclose the area is

$$F = 2x + y$$

where x is the dimension of the sides of the fence adjacent to the wall and y is the dimension of the side opposite the wall.

We can use the fact that the are needs to be $160{,}000$ square meters to write the amount of fencing needed in only one variable.

$$A = xy = 160{,}000$$

$$y = \frac{160{,}000}{x}$$

$$F = 2x + \frac{160{,}000}{x}$$

Now we can find the minimum amount of fencing by finding the derivative of the fencing equation and using it to find the critical numbers and then the minimum.

$$F' = 2 - \frac{160{,}000}{x^2}$$

$$2 - \frac{160{,}000}{x^2} = 0$$

$$2 = \frac{160{,}000}{x^2}$$

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$$2x^2 = 160{,}000$$

$$x^2 = 80{,}000$$

$$x = \pm\sqrt{80{,}000}$$

$$x \approx \pm 283$$

Since a negative amount of fencing doesn't make sense, $x \approx 283$ is our only critical number. We need to check that this is the minimum, which means we also need to check the endpoints.

Since one side of the fence would have to be at least one meter to contain a horse, a possible interval for the x variable could be $[1{,}160{,}000]$. Since $F(1) = 162{,}000$, $F(283) = 1{,}131$, and $F(160{,}000) = 320{,}001$, it's clear that the amount of fencing is minimum at the critical number $x \approx 283$. Then $y = 160{,}000/283 \approx 565$.

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Topic: Applied optimization

Question: What are the dimensions of the yard?

With $1{,}000$ m of new fencing material, you need to enclose all four sides of a square or rectangular yard, and you want to maximize the area of the yard.

Answer choices:

A $l = 250$ m and $w = 250$ m

B $l = 250$ m and $w = 750$ m

C $l = 750$ m and $w = 250$ m

D $l = 125$ m and $w = 375$ m

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Solution: A

As with any applied optimization problem, we need an optimization equation and a constraint equation.

Since we want to optimize (maximize) the area of the yard, the optimization equation will be an equation for its area, $A = lw$.

Since we've been told that we're using $1{,}000$ m of fencing, and that value isn't included in the optimization equation above, we know we'll need to use it in the constraint equation. In order to do that, we'll write an equation for the perimeter of the yard, $P = 2l + 2w$. We know the perimeter of the yard will be $1{,}000$ m, so we can say $1{,}000 = 2l + 2w$.

We have

Optimization	$A = lw$
Constraint	$1{,}000 = 2l + 2w$

We want to solve for a variable in the constraint equation so that we can use its value to plug into the optimization equation, thereby reducing the optimization equation to one variable only. Let's solve for w in the constraint equation.

$$2w = 1{,}000 - 2l$$

$$w = 500 - l$$

Plugging this value of w into our optimization equation, we get

$$A = l(500 - l)$$

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$$A = 500l - l^2$$

Now that our optimization equation is in one variable, this is just like a regular optimization equation where we find critical points and test them to see where we have a local maximum or local minimum. So we'll first take the derivative, then set it equal to 0, then solve for l to find critical points.

$$A' = 500 - 2l$$

$$500 - 2l = 0$$

$$2l = 500$$

$$l = 250$$

To make sure this critical point represents a maximum, we pick values on either side of $l = 250$, like $l = 240$ and $l = 260$ and plug them into the first derivative.

$$A'(240) = 500 - 2(240)$$

$$A'(240) = 500 - 480$$

$$A'(240) = 20$$

and

$$A'(260) = 500 - 2(260)$$

$$A'(260) = 500 - 520$$

$$A'(260) = -20$$

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Since $A'(240) = 20 > 0$, the function is increasing to the left of the critical point. And since $A'(260) = -20 < 0$, the function is decreasing to the right of the critical point. When the function goes up on the left and comes down on the right, the critical point represents a maximum, so we've proven that $l = 250$ is associated with a maximum.

We found the length l associated with the critical point, but we were asked for the dimensions that maximize area, so now we just need to find the width that corresponds with this length, which we'll do by plugging $l = 250$ into $w = 500 - l$.

$$w = 500 - 250$$

$$w = 250$$

We can conclude that the dimensions 250 m $\times$ 250 m are the dimensions that maximize the area of the yard, given that we're working with $1{,}000$ m of fencing.

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Topic: Applied optimization

Question: What is the volume of the box?

You want to construct a box with a square bottom and you only have 36 m^2 of material. Assuming you use all of the material, what is the maximum volume of the box?

Answer choices:

A $12\sqrt{6}$ m^3

B 36 m^3

C 18 m^3

D $6\sqrt{6}$ m^3

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Solution: D

As with any applied optimization problem, we need an optimization equation and a constraint equation.

Since we want to optimize (maximize) the volume of the box, the optimization equation will be an equation for its volume, $V = lwh$.

Since we've been told that the box has a square base, we know that $l = w$, so we can modify our volume equation to $V = l^2h$.

Since we've been told that we're using 36 m^2 of material, and that value isn't included in the optimization equation above, we know we'll need to use it in the constraint equation. In order to do that, we'll write an equation for the surface area of the box, $A = 2(wl) + 2(lh) + 2(wh)$. We know the surface area of the box will be 36 m^2 and we know that $l = w$, so we can say

$$36 = 2(wl) + 2(lh) + 2(wh)$$

$$18 = wl + lh + wh$$

$$18 = l^2 + lh + lh$$

$$18 = l^2 + 2lh$$

We have

Optimization	$V = l^2h$
Constraint	$18 = l^2 + 2lh$

We want to solve for a variable in the constraint equation so that we can use its value to plug into the optimization equation, thereby reducing the

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optimization equation to one variable only. Let's solve for h in the constraint equation.

$$2lh = 18 - l^2$$

$$h = \frac{18 - l^2}{2l}$$

Plugging this value of h into our optimization equation, we get

$$V = l^2 \frac{18 - l^2}{2l}$$

$$V = \frac{18l - l^3}{2}$$

$$V = 9l - \frac{1}{2}l^3$$

Now that our optimization equation is in one variable, this is just like a regular optimization equation where we find critical points and test them to see where we have a local maximum or local minimum. So we'll first take the derivative, then set it equal to 0, then solve for l to find critical points.

$$V' = 9 - \frac{3}{2}l^2$$

$$9 - \frac{3}{2}l^2 = 0$$

$$\frac{3}{2}l^2 = 9$$

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$$l^2 = 6$$

$$l = \pm\sqrt{6}$$

We can't have a negative length so we rule out $l = -\sqrt{6}$, and we know that the length associated with our critical point is $l = \sqrt{6}$. To make sure this critical point represents a maximum, we pick values on either side of $l = \sqrt{6}$, like $l = 2$ and $l = 3$ and plug them into the first derivative.

$$V'(2) = 9 - \frac{3}{2}(2)^2$$

$$V'(2) = 9 - 6$$

$$V'(2) = 3$$

and

$$V'(3) = 9 - \frac{3}{2}(3)^2$$

$$V'(3) = \frac{18}{2} - \frac{27}{2}$$

$$V'(3) = -\frac{9}{2}$$

Since $V'(2) = 3 > 0$, the function is increasing to the left of the critical point. And since $V'(3) = -9/2 < 0$, the function is decreasing to the right of the critical point. When the function goes up on the left and comes down on the right, the critical point represents a maximum, so we've proven that $l = \sqrt{6}$ is associated with a maximum.

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We found the length l associated with the critical point, but we were asked for the maximum volume, so now we just need to find the volume that corresponds with this length, which we'll do by plugging $l = \sqrt{6}$ into $V = 9l - (1/2)l^3$.

$$V = 9\sqrt{6} - \frac{1}{2}\left(\sqrt{6}\right)^3$$

$$V = 9\sqrt{6} - \frac{6}{2}\sqrt{6}$$

$$V = 9\sqrt{6} - 3\sqrt{6}$$

$$V = 6\sqrt{6}$$

The maximum volume the box can have is $6\sqrt{6}$ m^3.

We can conclude that $6\sqrt{6}$ m^3 is the maximum volume of the box, given that we're working with 36 m^2 of material and that base of the box is square.

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